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On boundary terms and conformal transformations in curved space-times

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Abstract

We intend to clarify the interplay between boundary terms and conformal transformations in scalar-tensor theories of gravity. We first consider the action for pure gravity in five dimensions and show that, on compactifing *a la* Kaluza-Klein to four dimensions, one obtains the correct boundary terms in the Jordan (or String) Frame form of the Brans-Dicke action. Further, we analyze how the boundary terms change under the conformal transformations which lead to the Pauli (or Einstein) frame and to the non-minimally coupled massless scalar field. In particular, we study the behaviour of the total energy in asymptotically flat space-times as it results from surface terms in the Hamiltonian formalism.

1 Introduction

It is known that there is a chain of conformal transformations which relates higher-dimensional pure gravity compactified to four dimensions in the so called Jordan (or String) frame (SF) to the Pauli (or Einstein) frame (EF) with a minimally coupled scalar field (for a review, see *e.g.*, Ref. [1]) and the latter to a non-minimally coupled scalar field [2] (and Refs. therein). This issue is however not free of controversy, the main points being the consistency of the original Kaluza-Klein (KK) compactification and which "frame" is to be taken in four dimensions as that possibly describing real-world Physics [2].

In the present paper we focus on another aspect which is usually "overlooked" in the literature, namely the role played by boundary terms in the various forms of the action and the way they change under such conformal transformations. For simplicity, we shall consider the case of five-dimensional pure gravity, for which there is no "ground state" problem (Minkowski space can be easily recovered when there is just one extra dimension). Our starting point is the Einstein-Hilbert action on the five-dimensional space-time manifold $\hat{\mathcal{M}}$,

$$\hat{S} = \frac{1}{16\,\pi\,\hat{G}} \int_{\hat{\mathcal{M}}} d^5x\,\sqrt{-\hat{g}}\,\hat{R}\ , \tag{1.1}$$

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where $\hat{G}$ is Newton constant in five dimensions, $\hat{g}_{AB}$ the five-dimensional metric ($A, B = 0, \ldots, 4$ and $\mu, \nu = 0, \ldots, 3$),

$$\hat{g}_{AB} = \begin{bmatrix} \bar{g}_{\mu\nu} + \kappa^2 \, \hat{\phi}^2 \, A_\mu \, A_\nu & \kappa \, \hat{\phi}^2 \, A_\nu \\ \kappa \, \hat{\phi}^2 \, A_\mu & \hat{\phi}^2 \end{bmatrix} , \tag{1.2}$$

and $\hat{R}$ its scalar curvature. Upon defining $\hat{\phi} = e^{-\phi}$ and compactifying *a la* KK [1] the metric and then setting the constant $\kappa = 0$ one can arrive at the Jordan (or String frame - SF) form of the Brans-Dicke action (without matter fields)

$$S_{SF} = \int_{\mathcal{M}} d^4x \, \sqrt{-\bar{g}} \, e^{-\phi} \left[\frac{\bar{R}}{16 \, \pi \, G} + \omega \, \left(\bar{\nabla}_\mu \phi \right) \, \left(\bar{\nabla}^\mu \phi \right) \right] . \tag{1.3}$$

In the above, the scalar field ϕ is also called the *dilaton* in string theory [3], $\bar{R}$ is the scalar curvature and $\bar{\nabla}$ the covariant derivative for the four-dimensional metric $\bar{g}_{\mu\nu}$, and the four-dimensional Newton constant $G = \hat{G}/V_S$, where $V_S = 2\,\pi$ is the "coordinate volume" of the circle S^1. We shall set $8\,\pi\,G = 1$ and $\omega = 1/4$ from now on, since the following considerations hold in general and any choice of these constants does not affect the analysis of boundary terms.

The conformal transformation

$$\bar{g}_{\mu\nu} = \Omega^2 \, g_{\mu\nu} \ , \tag{1.4}$$

with

$$\Omega = e^{\phi/2} \ , \tag{1.5}$$

yields the Pauli (or Einstein frame - EF) form of the action

$$S_{EF}[g_{\mu\nu}, \phi] = \frac{1}{2} \int_{\mathcal{M}} d^4x \, \sqrt{-g} \, \left[R - \left(\nabla_\mu \phi \right) \, \left(\nabla^\mu \phi \right) \right] \ , \tag{1.6}$$

in which ϕ is minimally-coupled to gravity and has zero conformal weight.

A non-minimal coupling (parameterized by the constant ξ) can finally be introduced by transforming the metric according to

$$\tilde{g}_{\mu\nu} = \Omega_\xi^2 \, g_{\mu\nu} \ , \tag{1.7}$$

where $\Omega_\xi^2 \equiv \left(1 - \xi \, \tilde{\phi}^2 \right)^{-1}$. The new scalar field is related to ϕ by

$$\frac{d\tilde{\phi}}{d\phi} = \frac{1 - \xi \, \tilde{\phi}^2}{\left[1 - \xi \, \left(1 - 6\,\xi \right) \, \tilde{\phi}^2 \right]^{1/2}} \ . \tag{1.8}$$

We note that $\tilde{\phi}$ cannot be assigned a specific conformal weight for generic ξ. Explicit solutions of Eq. (1.8) are rather involved, except for the conformally coupled case $\xi = 1/6$ (see Appendix A). In general the action reads

$$\tilde{S}[\tilde{g}_{\mu\nu}, \tilde{\phi}] = \frac{1}{2} \int_{\mathcal{M}} d^4x \, \sqrt{-\tilde{g}} \, \left[\left(1 - \xi \, \tilde{\phi}^2 \right) \, \tilde{R} - \left(\tilde{\nabla}_\mu \tilde{\phi} \right) \, \left(\tilde{\nabla}^\mu \tilde{\phi} \right) \right] \ , \tag{1.9}$$

[1]The basic assumptions are that the five-dimensional space-time has cylindrical symmetry, $\hat{\mathcal{M}} = \mathcal{M} \times S^1$, and all fields do not depend on the fifth dimension.

where $\tilde{\nabla}_\mu$ and $\tilde{R}$ represent the covariant derivative with respect to and the scalar curvature of the metric $\tilde{g}_{\mu\nu}$.

In the above manipulations we have always neglected all possible boundary terms at the border $\partial\mathcal{M}$ of $\mathcal{M}$ which arise from the scalar curvature. In fact, the latter changes under a conformal transformation (1.4) according to [4],

$$R \to \Omega^{-2}\left[R - 2\,(d-1)\,\Omega^{-1}\,\nabla_\mu\nabla^\mu\Omega - (d-1)\,(d-4)\,\Omega^{-2}\,\nabla_\mu\Omega\,\nabla^\mu\Omega\right]\ , \tag{1.10}$$

which holds in d space-time dimensions and for any Ω. However, such terms must be handled carefully [5, 6, 7] in order to obtain the correct equations of motion both in the Lagrangian and Hamiltonian formalism. This is what we shall deal with in the next Sections.

In Section 2 we consider the role of boundary terms in the Lagrangian formalism and, in Section 3, in the Hamiltonian formalism. The latter case is particularly interesting because the surface terms can therein be related to the value of the canonical Hamiltonian of the system, that is, the ADM mass in asymptotically flat space-times. Particular cases are then shown in Appendix B.2.

2 Boundary terms in the Lagrangian formalism

From the point of view of the Lagrangian formalism, the Einstein-Hilbert (EH) action,

$$S_{EH} = \frac{1}{2}\int_{\mathcal{M}} d^4x\,\sqrt{-g}\,R\ , \tag{2.1}$$

is ill-defined if the border of the space-time manifold, $\partial\mathcal{M}$, is not an empty set. In fact, the equations of motion in the Lagrangian formalism are obtained by varying the action with the value of the field variables held fixed at the boundary, namely

$$\delta g_{\mu\nu}|_{\partial\mathcal{M}} = 0\ . \tag{2.2}$$

However, the presence of second derivatives of the metric inside the scalar curvature R would also require the condition

$$\delta\partial_\lambda g_{\mu\nu}|_{\partial\mathcal{M}} = 0\ , \tag{2.3}$$

which can be disposed of by subtracting from the volume part of the EH action the surface term containing (minus) the (trace of the) extrinsic curvature K of the boundary [5, 7],

$$S_{EH} \to S'_{EH} = S_{EH} + \int_{\partial\mathcal{M}} d^3\zeta\,\sqrt{|\gamma|}\,K\ , \tag{2.4}$$

where γ_{ij} is the three-metric on $\partial\mathcal{M}$ and S'_{EH} contains no term linear in second derivatives of the metric.

It is straigtforward to generalize the above prescription to the five-dimensional action (1.1),

$$\hat{S} \to \hat{S}' = \hat{S} + \int_{\partial\hat{\mathcal{M}}} d^4\zeta\,\sqrt{|\gamma^{(4)}|}\,K^{(4)}\ , \tag{2.5}$$

where $\gamma^{(4)}_{\mu\nu}$ is the metric on the four-dimensional border $\partial\hat{\mathcal{M}}$ and $K^{(4)}$ the trace of its extrinsic curvature. After the dimensional reduction [2], from Eq. (2.5) one obtains

$$S'_{SF}[\bar{g}_{\mu\nu},\phi] = S_{SF}[\bar{g}_{\mu\nu},\phi] + \int_{\partial\mathcal{M}} d^3\zeta\,\sqrt{|\bar{\gamma}|}\,e^{-\phi}\,\bar{K}\ . \tag{2.6}$$

[2]We assume the border of the five-dimensional manifold also has cylindrical symmetry, $\partial\hat{\mathcal{M}} = \partial\mathcal{M}\times S^1$

We shall now show that, as one would expect, this is precisely the generalization of the prescription (2.4) which is required to have a well-defined Lagrangian form.

2.1 String Frame and Einstein Frame

Upon changing to the EF according to the transformation (1.5) one obtains

$$S_{SF}[\bar{g}_{\mu\nu},\phi] = S_{EF}[g_{\mu\nu},\phi] - \frac{3}{2}\int_{\mathcal{M}} d^4x\,\sqrt{-g}\,\left(\nabla_\mu\nabla^\mu\phi\right)\ . \tag{2.7}$$

One now notices that

$$\sqrt{|\bar{\gamma}|}\,e^{-\phi}\,\bar{K} = \sqrt{|\gamma|}\,K + \frac{3}{2}\sqrt{|\gamma|}\,n^\mu\,\nabla_\mu\phi\ , \tag{2.8}$$

where n^μ is the (time- or space-like) normal to $\partial\mathcal{M}$. One can therefore conclude that the second term in the right hand side (r.h.s.) of Eq. (2.7) is exactly cancelled against the second term above, since

$$\int_{\mathcal{M}} d^4x\,\sqrt{-g}\,\nabla_\mu\nabla^\mu\phi = \int_{\mathcal{M}} d^4x\,\partial_\mu\left(\sqrt{-g}\,\nabla^\mu\phi\right) = \int_{\partial\mathcal{M}} d^3\zeta\,\sqrt{|\gamma|}\,n^\mu\,\nabla_\mu\phi\ . \tag{2.9}$$

Hence

$$S'_{SF}[\bar{g}_{\mu\nu},\phi] = S_{EF}[g_{\mu\nu},\phi] + \int_{\partial\mathcal{M}} d^3\zeta\,\sqrt{|\gamma|}\,K \equiv S'_{EF}[g_{\mu\nu},\phi]\ . \tag{2.10}$$

In other words, if one takes properly into account the boundary terms in the action (2.6), no boundary terms are generated by the transformation (1.5) except those which are required to dispose of the (unwanted) condition on the first derivatives of the metric, such as that in Eq. (2.3).

2.2 Non-minimally coupled scalar field

The action for a non-minimally coupled, massless scalar field $\tilde{\phi}$ in a space-time $\mathcal{M}$ with metric $\tilde{g}_{\mu\nu}$ is given by Eq. (1.9) [4]. On carefully performing the transformations (1.7) and (1.8), one actually finds that a boundary term is also generated, so that

$$\begin{aligned}\tilde{S}[\tilde{g}_{\mu\nu},\tilde{\phi}] &= S_{EF}[g_{\mu\nu},\phi] - 3\int_{\mathcal{M}} d^4x\,\sqrt{-g}\,\left(\nabla_\mu\nabla^\mu\ln\Omega_\xi\right)\\ &= S_{EF}[g_{\mu\nu},\phi] - 3\int_{\partial\mathcal{M}} d^3\zeta\,\sqrt{|\gamma|}\,n^\mu\,\nabla_\mu\ln\Omega_\xi\ .\end{aligned} \tag{2.11}$$

In order to eliminate second derivatives of the metric, one can now employ the following prescription [8] instead of (2.4)

$$\tilde{S}[\tilde{\phi},\tilde{g}_{\mu\nu}] \to \tilde{S}'[\tilde{\phi},\tilde{g}_{\mu\nu}] = \tilde{S}[\tilde{\phi},\tilde{g}_{\mu\nu}] + \int_{\partial\mathcal{M}} d^3\zeta\,\sqrt{|\tilde{\gamma}|}\,\left(1-\xi\,\tilde{\phi}^2\right)\,\tilde{K}\ . \tag{2.12}$$

Since

$$\sqrt{|\tilde{\gamma}|}\,\left(1-\xi\,\tilde{\phi}^2\right)\,\tilde{K} = \sqrt{|\gamma|}\,K + 3\sqrt{|\gamma|}\,n^\mu\,\nabla_\mu\ln\Omega_\xi\ , \tag{2.13}$$

one again finds that

$$\tilde{S}'[\tilde{g}_{\mu\nu},\tilde{\phi}] = S_{EF}[g_{\mu\nu},\phi] + \int_{\partial\mathcal{M}} d^3\zeta\,\sqrt{|\gamma|}\,K \equiv S'[g_{\mu\nu},\phi]\ , \tag{2.14}$$

and concludes that unwanted boundary terms are not generated by the transformations (1.7) and (1.8), in complete analogy with the case of Section 2.1.

3 Boundary terms in the Hamiltonian formalism

When one moves on to the Hamiltonian formalism, the handling of surface terms becomes more subtle because they acquire a dynamical meaning. One first performs the ADM decomposition [9] of the metric,

$$g_{\mu\nu} = \left[\begin{array}{cc} -N^2 + N_k\, N^k & N_j \\ N_i & \gamma_{ij} \end{array} \right] , \tag{3.1}$$

where N is the lapse function and N^i are the shift functions associated to a given foliation of $\mathcal{M}$ into spatial hypersurfaces Σ_t whose unit time-like normal is denoted by

$$t^\mu = \left(\frac{1}{N} \, , \, -\frac{N^i}{N} \right) . \tag{3.2}$$

The EH Lagrangian can then be written as

$$L_{EH} = \frac{1}{2} \int_{\Sigma_t} d^3x \, \sqrt{-g}\, R \, , \tag{3.3}$$

with

$$\begin{aligned} \sqrt{-g}\, R \;\; &= \;\; N \sqrt{\gamma} \left(K_{ij}\, K^{ij} - K^2 + R^{(3)} \right) \\ & \quad\; -2 \left[\sqrt{\gamma}\, K \right]_{,t} + 2 \left[\sqrt{\gamma} \left(K\, N^j - \gamma^{ij}\, N_{,i} \right) \right]_{,j} \, , \end{aligned} \tag{3.4}$$

where

$$K_{ij} \;\; = \;\; \frac{1}{2\,N} \left(N_{i|j} + N_{j|i} - \gamma_{ij,t} \right) \, , \tag{3.5}$$

is the extrinsic curvature tensor of the hypersurfaces Σ_t, $|$ denotes the covariant derivative with respect to the three-metric γ_{ij} and $R^{(3)}$ is its intrinsic curvature. Second time derivatives of the three-metric γ_{ij} should again not appear in the action and this is accomplished by adding a surface term of the form in Eq. (2.4) on the initial (Σ_{t_1}) and final (Σ_{t_2}) slices,

$$\left[\int_{\Sigma_t} d^3x \, \sqrt{|\gamma|}\, K \right]_{t_1}^{t_2} = \int_{t_1}^{t_2} dt \int_{\Sigma_t} d^3x \, \left[\sqrt{\gamma}\, K \right]_{,t} \, , \tag{3.6}$$

which cancels against the second term in the r.h.s. of Eq. (3.4) above. However, one finds that the surface terms at $\partial\Sigma_t$ which arise from $R^{(3)}$ and the third term in Eq. (3.4) are related to the energy and momentum of the system, as we review below.

Since we are mainly concerned with symptotically flat space-times, from now on we assume that the border of Σ_t is a two-sphere and introduce (asymptotically, *i.e.*, at $r \to \infty$) spherical coordinates (r, θ, φ) such that $\partial\Sigma_t \equiv \{r = R\}$. It is then consistent to impose the following conditions [6]

$$\begin{aligned} & \gamma_{ij} - \eta_{ij} \sim N - 1 \sim N^i \sim r^{-1} \\ & \gamma_{ij,k} \sim N_{,k} \sim N^i_{,k} \sim r^{-2} \, . \end{aligned} \tag{3.7}$$

One can thus neglect terms containing the N^i's and, after adding the term in Eq. (3.6) above, one obtains the Lagrangian [3]

$$L_{EH}^{(0)} = \frac{1}{2}\int_{\Sigma_t} d^3x\, N\,\sqrt{\gamma}\,\left(K_{ij}\,K^{ij} - K^2 + R^{(3)}\right) - \int d\theta\, d\varphi\, \left[\sqrt{\gamma}\,\gamma^{ri}\,N_{,i}\right]_{r=R} . \tag{3.8}$$

Since $L_{EH}^{(0)}$ still contains second (spatial) derivatives, it does not meet the prescription required by the Lagrangian formalism and, as a consequence, upon varying the Hamiltonian ($\pi^{ij} \equiv \frac{\delta L_{EF}^{(0)}}{\delta\gamma_{ij,t}}$ are the canonical momenta conjugated to γ_{ij})

$$H_{EH}^{(0)} = \int_{\Sigma_t} d^3x\,\left(\pi^{ij}\,\gamma_{ij,t}\right) - L_{EH}^{(0)} , \tag{3.9}$$

one gets unwanted terms in the Hamiltonian form of the field equations [6]. Such terms can be eliminated by adding a suitable surface term to the Hamiltonian $H_{EF}^{(0)}$, for which just the asymptotic form is usually given [10, 6, 7].

For the sake of simplicity, we assume for the three-metric the form

$$\gamma_{ij}\,dx^i\,dx^j = f^2\,dr^2 + h^2\,\left(d\theta^2 + \sin^2\theta\,d\varphi^2\right) , \tag{3.10}$$

where $f = f(r)$ and $h = h(r)$, and neglect the N^i's in agreement with the conditions in Eq. (3.7) and spherical symmetry. The three-curvature thus gives rise to a term

$$N\,\sqrt{\gamma}\,R^{(3)} \supseteq -N\,f\,h^2\,\sin\theta\,\frac{4\,h_{,rr}}{h\,f^2} = -\left[4\,N\,\sin\theta\,\frac{h\,h_{,r}}{f}\right]_{,r} + \dots , \tag{3.11}$$

which, once added to the second term in the r.h.s. of Eq. (3.8), yields

$$-\int d(\cos\theta)\,d\varphi\,\left(r^2\,N\,N_{,r} + 2\,r\,N^2\right)_{r=R} = -\int_{\partial\Sigma_t} d\theta\,d\varphi\,\sqrt{\gamma_R}\,K_R \equiv E_{EH} . \tag{3.12}$$

In Eq. (3.12) we have set

$$f = N^{-1} , \qquad h = r , \tag{3.13}$$

and γ_R is the three-metric of the surface $r = R$ with K_R the corresponding extrinsic curvature.

Unwanted terms can now be eliminated from the Hamiltonian by subtracting from $L_{EH}^{(0)}$ the usual boundary term

$$L'_{EH} = L_{EH}^{(0)} - E_{EH} . \tag{3.14}$$

from which one recovers the expressions given in [10, 6, 7] for $R \to \infty$.

It is now interesting to recall that L'_{EH} reduces to $-E_{EH}$ when it is evaluated for a (static) solution of the field equations. This is not surprising, since any solution satisfies the constraints corresponding to the reparameterization invariances of General Relativity,

$$\mathcal{H} = \mathcal{H}_i = 0 , \tag{3.15}$$

where $\mathcal{H}$ and $\mathcal{H}_i$ are implicitly defined by

$$H_{EH}^{(0)} = \int_{\Sigma_t} d^3x\,\left(N\,\mathcal{H} + N^i\,\mathcal{H}_i\right) . \tag{3.16}$$

For a static space-time $\gamma_{ij,t} = 0$, thus the above constraints imply that $H_{EH}^{(0)} = L_{EH}^{(0)} = 0$ [4].

[3] Generalization to the case when there is a matter source is straightforwardly obtained by adding the matter Lagrangian, $L_{EH}^{(0)} \to L_{EH}^{(0)} + L_M$.

[4] In general $L^{(0)} = 0$ for a static source (see footnote 3).

Therefore, the total Hamiltonian becomes

$$H'_{EH} \equiv H^{(0)}_{EH} + E_{EH} = E_{EH} \ , \tag{3.17}$$

and is proportional to the ADM mass M contained within the sphere of radius R in the limit $R \to \infty$ (provided one subtracts a flat space contribution [5], see also Appendix B.2)

Under a general conformal transformation (1.7) one expects that the value of the energy as measured locally changes, since such a transformation represents a (local) rescaling of the units [11] according to the relation

$$\bar{E} = \Omega^{-1}\, E \ , \tag{3.18}$$

which follows from naive dimensional analysis. However, the same argument does not apply to the ADM mass, since M is a global quantity related to the total energy contained in the system. In fact, one can consider the geodesic equations at large r in the Newtonian limit and for a static space-time in the rescaled frame one obtains

$$\begin{aligned} \frac{d^2r}{dt^2} &\simeq \frac{1}{2}\,\bar{g}^{rr}\,\partial_r\bar{g}_{tt} \\ &= \frac{1}{2}\,g^{rr}\,g_{tt,r} + g^{rr}\,g_{tt}\,\partial_r(\ln\Omega) \ , \end{aligned} \tag{3.19}$$

where the first term in the r.h.s. above is the unrescaled frame contribution and the second term thus represents a correction. If Ω contains a term of order r^{-1},

$$\Omega \sim 1 - \frac{\delta M}{8\,\pi\,r} \ , \tag{3.20}$$

where δM is a constant (not necessarily small), and N is as in Eq. (B.8), then Eq. (3.19) yields

$$\frac{d^2r}{dt^2} \simeq -\frac{\bar{M}}{8\,\pi\,r^2} \ , \tag{3.21}$$

where $\bar{M} = M + \delta M$. However, upon substituting the form (3.20) into Eq. (3.18), one obtains $\bar{M} = M$, in contrast with what one finds, for instance, in the particular case of spherically symmetric dilatonic black holes [12] (see Appendix B.2). This consideration supports the argument put forward in Ref. [12] that different conformal frames can be distinguished by measuring the ADM mass.

As in Section 2 we now analyze the effect of conformal transformations on the boundary terms.

3.1 String Frame and Einstein frame

Adding a minimally-coupled scalar field does not change the gravitational part of the action, therefore one finds $E_{EH}[\gamma_{ij}]$ in the corresponding total Hamiltonian H'_{EF}. The ADM decomposition is again understood and it is known that the conformal rescaling (1.5) is then a canonical transformation [13] provided second time derivatives of the three-metric are eliminated from the action. One is then left with a surface term on $\partial\Sigma_t$ which yields $E_{EF}[\gamma_{ij}] = E_{EH}[\gamma_{ij}]$.

The action S'_{EF} then transforms to the SF according to Eq. (2.10), which we can now rewrite as

$$S'_{SF}[\bar{g}_{\mu\nu},\phi] = S_{SF}[\bar{g}_{\mu\nu},\phi] - \int_{t_1}^{t_2} dt\, E_{EH}[\bar{g}_{\mu\nu}] \ , \tag{3.22}$$

and one obtains

$$E_{SF}[\bar{g}_{\mu\nu}, \phi] = E_{EH}[\bar{g}_{\mu\nu}] \ . \tag{3.23}$$

This proves that the ADM mass can be computed by simply replacing the EF metric with the SF metric inside Eq. (3.12).

In Appendix B.2 we show that for dilatonic black holes, since $\Omega = e^{\phi/2}$ is of the form in Eq. (3.20), one measures a different ADM mass in the SF with respect to the EF [12].

3.2 Non-minimally coupled scalar field

Under the conformal transformation defined by Eqs. (1.7) and (1.8), the EF action with a minimally coupled scalar field transforms according to Eq. (2.14), and yields

$$\tilde{S}'[\tilde{g}_{\mu\nu}, \tilde{\phi}] = \tilde{S}[\tilde{g}_{\mu\nu}, \tilde{\phi}] - \int_{t_1}^{t_2} dt\, E_{EH}[\tilde{g}_{\mu\nu}] \ . \tag{3.24}$$

Again, one finds that

$$\tilde{E}[\tilde{g}_{\mu\nu}, \tilde{\phi}] = E_{EH}[\tilde{g}_{\mu\nu}] \ . \tag{3.25}$$

In general one expects that $\phi \sim r^{-1}$, therefore Ω_ξ does not contain terms of order r^{-1} and the ADM mass is not changed by the transformation in Eqs. (1.7) and (1.8).

4 Conclusions

In this paper we have analyzed in detail the surface terms which arise when performing certain conformal transformations that relate the SF to the EF and the latter with a minimally coupled scalar field to a non-minimally coupled one. We showed that such transformations do not give rise to boundary terms, except those which are needed to eliminate second derivatives.

Boundary terms must be handled with particular care in the canonical formalism, since they are related to the total energy of a system in asymptotically flat space. We have seen that the values taken by such terms generally depend on the conformal frame and, in particular, the ADM mass of dilatonic black holes in the EF is different from the one in the SF. This fact is not surprising (see Appendix B.2 and Ref. [12]), however, it conspires to question which frame is physical [2, 12].

Finally, despite the fact that our analysis was purely classical, one might consider the effect of surface terms in a quantum context and argue that they can be related to different sectors of the Hilbert space of states (see Ref. [14] for analogous considerations). We wish to return to this point in the future.

A Conformally coupled case

The general relation (1.8) between ϕ and $\tilde{\phi}$ is rather involved, however, the special case of conformal coupling ($\xi = 1/6$) allows the simple expressions [15]

$$\tilde{\phi} = \tanh\phi \qquad \text{or} \qquad \tilde{\phi} = \mathrm{cotanh}\,\phi \ . \tag{A.1}$$

Correspondingly one has

$$\Omega_{1/6} = \frac{1}{\cosh\phi} \qquad \text{or} \qquad \Omega_{1/6} = -\frac{i}{\sinh\phi} \ . \tag{A.2}$$

In asymptotically flat spaces, $\phi \simeq C\, r^{-1}$ for large r and only the first case in Eq. (A.2) is acceptable, with

$$\Omega_{1/6} \simeq 1 + \frac{C^2}{r^2} \ . \tag{A.3}$$

Since there is no term of order r^{-1}, the ADM mass does not depend on the conformal frame, as was already pointed out at the end of Section 3.2.

B Examples

In this Appendix we consider two cases with the purpose of showing how surface terms in the Hamiltonian formalism behave. In the first case (Section B.1) we just have second time derivatives of the three-metric which must be eliminated by adding a surface term along the hypersurfaces of initial and final time; in the second case (Section B.2), instead, we analyze surface terms at large distance from a central source.

B.1 Friedmann-Robertson-Walker space-time

For the case of a non-minimally coupled scalar field in (spatially-flat) FRW space-time, the action (1.9) for $\tilde{a} = \tilde{a}(t)$ (scale factor of the Universe) and $\tilde{\phi} = \tilde{\phi}(t)$ reduces to

$$\tilde{S} = \frac{1}{2} \int_{t_1}^{t_2} dt\, \tilde{N}\, \tilde{a}^3 \left\{ \frac{\dot{\tilde{\phi}}^2}{\tilde{N}^2} + \frac{6}{\tilde{a}^2} \left(1 - \xi\, \tilde{\phi}^2\right) \left[\frac{\dot{\tilde{a}}^2}{\tilde{N}^2} + \frac{\tilde{a}}{\tilde{N}} \frac{d}{dt} \left(\frac{\dot{\tilde{a}}}{\tilde{N}} \right) \right] \right\} \ , \tag{B.1}$$

in which the shift functions $\tilde{N}^i$ have been set to zero and $\tilde{N} = \tilde{N}(t)$. Upon getting rid of second time derivatives, according to the prescription (2.12) with

$$\sqrt{\tilde{\gamma}}\, \tilde{K} = -3\, \tilde{a}^2\, \frac{\dot{\tilde{a}}}{\tilde{N}} \ , \tag{B.2}$$

one obtains [8]

$$\tilde{S}' = \frac{1}{2} \int_{t_1}^{t_2} dt\, \tilde{N}\, \tilde{a}^3 \left[\frac{\dot{\tilde{\phi}}^2}{\tilde{N}^2} - \frac{6}{\tilde{a}^2} \left(1 - \xi\, \tilde{\phi}^2\right) \frac{\dot{\tilde{a}}^2}{\tilde{N}^2} + 12\, \xi\, \frac{\tilde{\phi}}{\tilde{a}}\, \frac{\dot{\tilde{a}}\, \dot{\tilde{\phi}}}{\tilde{N}^2} \right] \ . \tag{B.3}$$

The conformal transformation of the metric defined by Eq. (1.7) then becomes

$$\tilde{N} = \Omega_\xi\, N \ , \qquad \tilde{a} = \Omega_\xi\, a \ , \tag{B.4}$$

and yields the action

$$S'_{EF} = \frac{1}{2} \int dt\, N\, a^3 \left[\frac{\dot{\phi}^2}{N^2} - \frac{6\, \dot{a}^2}{a^2\, N^2} \right] \ , \tag{B.5}$$

in which no second time derivatives appear as well.

The same form as in Eq. (B.5) can be arrived at from the SF action

$$S_{SF} = \frac{1}{2} \int dt\, \bar{N}\, \bar{a}^3\, e^{-\phi} \left[\frac{6\, \dot{\bar{a}}^2}{\bar{a}^2\, \bar{N}^2} + \frac{6}{\bar{a}\, \bar{N}} \frac{d}{dt} \left(\frac{\dot{\bar{a}}}{\bar{N}} \right) - \frac{\dot{\phi}^2}{2\, \bar{N}^2} \right] , \tag{B.6}$$

by first subtracting second time derivatives,

$$S'_{SF} = \frac{1}{2} \int dt\, \bar{N}\, \bar{a}^3\, e^{-\phi} \left[\frac{6\, \dot{\bar{a}}\, \dot{\phi}}{\bar{a}\, \bar{N}^2} - \frac{6\, \dot{\bar{a}}^2}{\bar{a}^2\, \bar{N}^2} - \frac{\dot{\phi}^2}{2\, \bar{N}^2} \right] , \tag{B.7}$$

and then defining

$$\bar{N} = e^{\phi/2}\, N \ , \qquad \bar{a} = e^{\phi/2}\, a \ .$$

B.2 Black holes

The classical example which is used to show the role played by boundary terms in the canonical formalism is given by the Schwarzschild metric, which is of the form given by Eqs. (3.10) and (3.13) with

$$N^2 = 1 - \frac{M}{4\, \pi\, r} \ . \tag{B.8}$$

Any metric satisfying the conditions (3.7) approaches the above form for $r \to \infty$ and one then finds

$$\begin{aligned} E_{EH} &= -\int d\theta\, d\varphi \left(r^2\, N\, N' + 2\, r\, N^2 \right)_{r=R} \\ &\simeq \left[-\frac{M}{2} + (2\, M - 16\, \pi\, r) \right]_{r=R\gg M} \\ &= \frac{3}{2}\, M - 16\, \pi\, R \ , \end{aligned} \tag{B.9}$$

where M is the ADM mass of the black hole. The diverging term (for $R \to \infty$) in the r.h.s. of the above equation can be eliminated by subtractiong the flat space contribution [5]

$$\begin{aligned} E_{EH}[g_{\mu\nu}] \to E_{EH}[g_{\mu\nu}] - E_{EH}[\eta_{\mu\nu}] &= -\int d\theta\, d\varphi \left[\sqrt{\gamma_R}\, K_R - r^2\, \sin\theta\, \frac{4}{r} \right]_{r=R\gg M} \\ &= \frac{3}{2}\, M \ . \end{aligned} \tag{B.10}$$

A less trivial example is given by the spherically symmetric charged black hole [16, 17], whose metric in the EF is given by [5]

$$ds^2 = -\left(1 - \frac{r_+}{r} \right) dt^2 + \left(1 - \frac{r_+}{r} \right)^{-1} dr^2 + r^2 \left(1 - \frac{r_-}{r} \right) d\Omega^2 \ , \tag{B.11}$$

where

$$\begin{aligned} r_+ &= \frac{M}{4\, \pi} \\ r_- &= 2\, \frac{Q^2}{r_+} \ , \end{aligned} \tag{B.12}$$

[5]The case shown here corresponds to the choice $a = 1$ for the dilaton coupling (see Ref. [12] for more details).

and the dilaton field is

$$e^{\phi} = 1 - \frac{r_-}{r} \ . \tag{B.13}$$

The ADM mass computed from the large r expansion of the metric is thus given by M and coincides with the quantity obtained from Eq. (3.12),

$$M = \frac{2}{3} E_{EH}[g_{\mu\nu}] = 4\,\pi\, r_+ \ , \tag{B.14}$$

Changing to the SF yields a different ADM mass, namely

$$\bar{M} = M + \frac{Q^2}{2\,M} \ , \tag{B.15}$$

which is again obtained both from the large r expansion of

$$\bar{g}_{tt} = -\left(1 - \frac{r_+}{r}\right)\left(1 - \frac{r_-}{r}\right) \ , \tag{B.16}$$

and Eq. (3.23),

$$\bar{M} = \frac{2}{3} E_{EH}[\bar{g}_{\mu\nu}] = 4\,\pi\,\left(r_+ + r_-\right) \ . \tag{B.17}$$

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